



Problem 1. Twelve different two-digit numbers are given. Prove that the difference between some two of them is a two-digit number in the form of aa

Source: Matematyka Olimpijska, Kombinatoryka, A. Neugebauer

Choice of problem: Maria Janyska

Solution: Let us note that all two-digit numbers of the form aa can be written as $11 \cdot a$. Therefore, we must show that one of the differences is divisible by 11 and is a two-digit number.

There are 11 different remainders modulo 11, so at least two of the twelve given numbers must have the same remainder modulo 11. Thus, their difference will be congruent to 0 (mod 11).

We know that the numbers are different, so the considered difference is not 0 — it is at least 11. Furthermore, since the numbers are two-digit numbers, their difference has at most 2 digits. This means it is a two-digit number divisible by 11, that is, it has the form aa .



Problem 2. Let $n \in \mathbb{Z}_+$. Show that the number $2(n^2 + 1) - n$ is never the square of an integer.

Source: CPSJ 2012, zawody drużynowe

Choice of problem: Maria Janyska

Rozwiązanie: Suppose, for the sake of contradiction, that this number is a square of an integer. Then for some $k \in \mathbb{Z}_+$, we have:

$$2(n^2 + 1) - n = k^2.$$

Multiply both sides of the equation by 8 and complete the square on the left-hand side:

$$(4n - 1)^2 + 15 = 8k^2.$$

Since $3|15$, $(4n - 1)^2 \equiv 8k^2 \pmod{3}$. Now, the square of any integer modulo 3 is congruent to 0 or 1. But the right-hand side, is congruent to 0 or 2 $\pmod{3}$. This implies both sides must be divisible by 3, so, as squares with all factors being even: $9|(4n - 1)^2, 9|k^2$ (since 8 is coprime to 3).

However, $15 \equiv 6 \pmod{9}$, and so:

$$(4n - 1)^2 + 15 \equiv 0 + 6 = 6 \pmod{9} \not\equiv 0 \equiv 8k^2.$$

Which is a contradiction. Therefore, the given number can never be a perfect square of an integer, as required.