Problem 1. Prove that the following inequality holds for every $x > 0$:

$$\sqrt{x+1} + \sqrt{x+5} \leqslant 2\sqrt{x+3}.$$

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Solution: Let $a = x+1$ and $b = x+5$. Substitute into the left-hand side of the inequality:

$$\sqrt{x+1} + \sqrt{x+5} = \sqrt{a} + \sqrt{b}.$$

Notice also that

$$\frac{a+b}{2} = \frac{x+1+x+5}{2} = x+3.$$

Since $x > 0$, we have $x+5 > x+1 > 0$, thus $(\sqrt{a})^2 = a$ and $\left(\sqrt{b}\right)^2 = b$. Therefore, our inequality becomes

$$\sqrt{a} + \sqrt{b} \leqslant 2\sqrt{\frac{(\sqrt{a})^2 + \left(\sqrt{b}\right)^2}{2}},$$

which holds by the inequality between the arithmetic mean and the quadratic mean. ■

For reference:

Twierdzenie 1 (Inequality between arithmetic and quadratic mean)

For any $n \in \mathbb{Z}_+$ and $a_1, a_2, \ldots, a_n > 0$ we have

$$\sqrt{\frac{a_1^2 + a_2^2 + \ldots + a_n^2}{n}} \geqslant \frac{a_1 + a_2 + \ldots + a_n}{n},$$

with equality if and only if

$$a_1 = a_2 = \ldots = a_n.$$

Problem 2. Let M be any point on the altitude AD of triangle ABC. The line parallel to AB passing through M intersects the altitude CF at P. The line parallel to AC passing through M intersects the altitude BE at Q. Prove that the lines PQ and EF are parallel.

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Solution: Let H be the orthocenter of ABC. The angles $\sphericalangle MPH$ and $\sphericalangle MQH$ are right, hence points M, P, Q, H lie on the circle with diameter MH. The angles $\sphericalangle AEH$ and $\sphericalangle AFH$ are right, hence points A, E, F, H lie on the circle with diameter AH. Therefore,

$$\sphericalangle MPQ = \sphericalangle MHQ = \sphericalangle AHE = \sphericalangle AFE,$$

which completes the proof, since $AF \parallel MP$. ■