Problem 1. Let the quadrilateral $ABDC$ be inscribed in a circle, and let X be the intersection point of its diagonals. Prove that

$$AB \cdot AC < AX \cdot (BC + AD).$$

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Solution: From the triangle inequality we know that

$$AB < AX + XB \qquad \text{and} \qquad AC < AX + XC.$$

Multiplying these inequalities (all terms are positive) gives

$$AB \cdot AC < (AX + XB) \cdot (AX + XC) = AX^2 + AX \cdot XB + AX \cdot XC + BX \cdot XC.$$

Let's recall the concept of *power of a point*:

Definicja 1 (Power of a Point)

The *power of a point* A with respect to a circle o with center O and radius r is defined as

$$P(A, o) = |AO|^2 - r^2,$$

where $|AO|$ is the distance from A to the center O of the circle.

From this definition, we can derive interesting properties:

Twierdzenie 1 (Properties of the Power of a Point)

- $P(A, o) > 0$ if A lies outside the circle,
- $P(A, o) = 0$ if A lies on the circle,
- $P(A, o) < 0$ if A lies inside the circle.

Twierdzenie 2 (Power of a Point Formula)

Let A be any point, and k a line passing through A intersecting the circle o at points B and C. Then

$$P(A, o) = |AB| \cdot |AC|.$$

If A lies outside the circle, then $P(A, o) = |AB| \cdot |AC|$; if A lies inside, then $P(A, o) = -|AB| \cdot |AC|$.
If D is the point of tangency of k with the circle, then

$$P(A, o) = |AD|^2.$$

Thus, in our problem,

$$AX \cdot XD = BX \cdot XC.$$

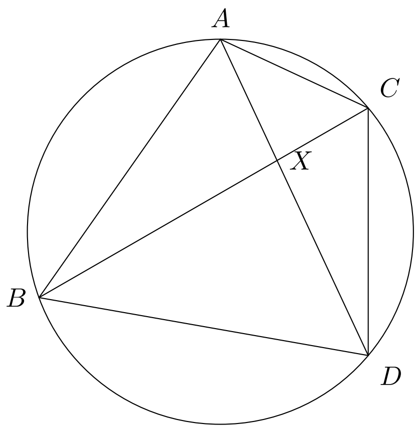
Substituting this into the inequality:

$$AX^2 + AX \cdot XB + AX \cdot XC + BX \cdot XC = AX^2 + AX \cdot XB + AX \cdot XC + AX \cdot XD,$$

so

$$AB \cdot AC < AX \cdot (AX + XD + BX + XC) = AX \cdot (AD + BC),$$

which completes the proof. ■



Zadanie 2. An infinite sequence $a_1, a_2, \ldots$ of positive integers will be called *bulldozer* if it satisfies the condition

$$a_i \mid a_j \iff j \mid i$$

for every pair of positive integers (i, j). Determine whether a bulldozer sequence exists.

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Solution: Assume that (a_n) is a bulldozer sequence. Note that $i \mid 2i$ for every i, hence

$$\cdots \mid a_4 \mid a_2 \mid a_1 \implies a_1 \geqslant a_2 \geqslant a_4 \geqslant \ldots$$

Since $a_i \geqslant 1$, the subsequence $a_1, a_2, a_4, a_8, \ldots$ eventually becomes constant. Let λ be such that $a_{2^\lambda} = a_{2^{\lambda+1}}$. Observe that

$$2^\lambda \mid 3 \cdot 2^\lambda \implies a_{3 \cdot 2^\lambda} \mid a_{2^\lambda} = a_{2^{\lambda+1}} \implies 2^{\lambda+1} \mid 3 \cdot 2^\lambda,$$

which is a contradiction. ■