Solutions to the Problems
from 08/25/2025

Problem 1. We consider integers modulo n, where $n > 1$. The inverse of an integer $x \pmod{n}$ is an integer y if

$$x \cdot y \equiv 1 \pmod{n}.$$

Prove that the number x has an inverse modulo n if and only if

$$\gcd(n, x) = 1.$$

Source: Magdalena Ciochoń, „Congruences", contest work written under the supervision of mgr Agnieszka Batko
Choice: Robert Rośczak
Solution: We will prove the implication in both directions:

$$\text{The number has an inverse} \implies \gcd(n, x) = 1.$$

Assume, for contradiction, that the number x has an inverse y, i.e. $x \cdot y = 1 \pmod{n}$, but $\gcd(n, x) > 1$. We will derive a contradiction.

Let $\gcd(n, x) = d > 1$ and $x \cdot y = kn + 1$, $k \in \mathbb{Z}_+$. Then clearly $d \mid n$, $d \mid kn$. From the Euclidean algorithm we can conclude that $\gcd(kn, kn+1) = \gcd(kn, 1) = 1$, hence for the chosen $d > 1$ it follows that $d \nmid kn + 1$. However, we know that $kn + 1 = x \cdot y$ and $d \mid x$, which gives us the desired contradiction with the uniqueness of prime factorization on both sides of the equality. Thus the implication in the first direction holds.

Now let us examine the implication in the other direction:

$$\gcd(n, x) = 1 \implies \text{the number has an inverse.}$$

Since $\gcd(n, x) = 1$, for $n > 1$ we know that $n \nmid x$, i.e. $x \not\equiv 0 \pmod{n}$.

Let us consider all possible values of $y \pmod{n}$. We have possible products $x \cdot 0, x \cdot 1, x \cdot 2, \ldots, x \cdot (n-1)$.

No two different products can yield the same remainder modulo n, because if at least two were congruent to the same number, their difference (a nonzero multiple of x) would be congruent to $0 \pmod{n}$, which contradicts the assumption that x

and n are coprime. Therefore, all n products give different remainders modulo n, and in particular one of them is congruent to 1. The value by which we multiply x in this case is precisely its inverse. Hence the implication in the second direction also holds.

Showing both directions gives us equivalence, which completes the proof of the problem.

Problem 2. Marek and Darek each have two identical bottles, partially filled with juice. It is known that if Darek had twice as much juice, he would have more than if Marek received an additional full bottle.

Let m denote the amount of juice Marek has minus the amount contained in one bottle. Similarly, define d for Darek.

Find the smallest possible value of the expression

$$2d^3 + 3d^2m - m^3.$$

Author: Robert Rośczak
Solution: Let the capacity of one bottle be B, and the amounts of juice held by Marek and Darek be M and D respectively. By definition we have

$$m = M - B, \qquad d = D - B.$$

The condition of the problem:

$$2D > M + B \quad \iff \quad 2(d + B) > (m + B) + B \quad \iff \quad 2d > m.$$

Let us transform the expression:

$$2d^3 + 3d^2m - m^3 = (2d - m)(d^2 + 2dm + m^2) = (2d - m)(d + m)^2.$$

From the inequality $2d > m$ we obtain $(2d - m) > 0$, while $(d + m)^2 \geqslant 0$ always. Thus

$$2d^3 + 3d^2m - m^3 \geqslant 0,$$

with equality if and only if $d + m = 0$, i.e. $M + D = 2B$. This is possible, for example, if Darek has two bottles, each filled to $\frac{3}{4}$, and Marek has two bottles each filled to $\frac{1}{4}$, which satisfies the inequality condition.