
Problem 1. Given is a circle o_1 with center O and its diameter MN. Point P lies on this circle, and the line NP intersects at point Q the line k perpendicular to the diameter MN passing through the center of the circle. Point P' is the reflection of point P with respect to line k, and point Q' is the intersection of line NP' with line k. Knowing that the radius of circle o_1 has length r, determine the product $OQ' \cdot OQ$ in terms of the radius length.

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Solution: We will show that the desired product equals R^2.
Since MN is a diameter, the angles subtended by it are right angles, including $\sphericalangle MPN$. Moreover, by symmetry in the circle o_1, the line MP passes through the point Q', so the points P, Q', M are collinear. One can transfer along the arc MP' the angles: $\sphericalangle ONQ' = \sphericalangle MNP' = \sphericalangle MPP'$, whereas $\triangle Q'PQ$ is a right triangle with the altitude dropped from the vertex at the right angle PP', which gives us the equality of angles $\sphericalangle MPP' = \sphericalangle Q'PP' = \sphericalangle PQQ'$.

Hence, we have the angle equalities $\sphericalangle ONQ' = \sphericalangle OQN$ and the common right angle $\sphericalangle NOQ$ for $\triangle Q'ON$ and $\triangle NOQ$, which implies the similarity of these triangles by angle equality. Therefore, we can write down the corresponding ratios of the bases:

$$\frac{ON}{OQ'} = \frac{OQ}{ON}$$
$$OQ' \cdot OQ = ON \cdot ON = R^2,$$

which is what we wanted to compute.

Problem 2. Given a positive integer n. Prove that there exists a circle in the coordinate plane that contains exactly n lattice points (that is, points with both coordinates integers) in its interior.

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Solution: Any two distinct lattice points $\mathbb{Z}^2$ have different distances from the point $w = \left(\sqrt{2},\, \frac{1}{3}\right)$, i.e., there is no circle with this center passing through two or more lattice points.

Indeed, let $a = (a_x,\, a_y)$, $b = (b_x,\, b_y) \in \mathbb{Z}^2$ with $a \neq b$. If $|a - w| = |b - w|$, then

$$\left(a_x - \sqrt{2}\right)^2 + \left(a_y - \tfrac{1}{3}\right)^2 = \left(b_x - \sqrt{2}\right)^2 + \left(b_y - \tfrac{1}{3}\right)^2,$$

that is,

$$a_x^2 - a_y^2 - b_x^2 - b_y^2 - \tfrac{2}{3}\left(a_y - b_y\right) = 2\left(a_x - b_x\right)\sqrt{2}.$$

The right-hand side is thus a rational number, so $a_x = b_x$, but then

$$a_y^2 - b_y^2 - \tfrac{2}{3}\left(a_y - b_y\right) = \left(a_y - b_y\right)\left(a_y + b_y - \tfrac{2}{3}\right) = 0.$$

This is possible only when $a_y = b_y$. Hence $a = b$, a contradiction.

The lattice $\mathbb{Z}^2$ is a countable set, so using the above observation, we can arrange all of its elements in a sequence $\mathbb{Z}^2 = \{a_1,\, a_2,\, a_3,\, ...\}$ such that $|a_i - w| < |a_{i+1} - w|$ for $i = 1, 2, \ldots$ Then the open disk

$$x \in \mathbb{R}^2 : |x - w| < |a_{n+1} - w|$$

contains all lattice points $a_1, a_2, \ldots, a_n$ and no others.