Problem 1. Determine all positive integers n such that the number

$$\frac{2^n}{n+1}$$

is a square of an integer.

Bonus: Let A be the set of all positive integers $n > 2025$ such that the number

$$\frac{2^n}{n+1}$$

is a square of an integer, and let B be the set of all positive integers $n > 2025$ such that the number

$$\frac{2^{2^n}}{n+1}$$

is a square of an integer. Prove that the sets A and B are disjoint.

Author: Bartosz Trojanowski
Solution: Since the number $\frac{2^n}{n+1}$ is an integer, we have $n+1 \mid 2^n$. Hence, $n+1$ must be a power of two. Let $n+1 = 2^x$. Then

$$\frac{2^n}{n+1} = \frac{2^{2^x-1}}{2^x} = 2^{2^x-x-1}.$$

This number is a perfect square only if $2^x - x - 1$ is even and nonnegative. It follows that x must be odd. Moreover, $2^x \geqslant x+1$ for $x > 0$, so the numbers in question are of the form $n = 2^{2k+1} - 1$, where $k \geqslant 0$ is any integer. ■

Bonus solution: If there exists an n such that both $\frac{2^n}{n+1}$ and $\frac{2^{2^n}}{n+1}$ are perfect squares, then their product is also a perfect square. It is equal to

$$\frac{2^n \cdot 2^{2^n}}{(n+1)^2} = \frac{2^{2^n+n}}{(n+1)^2}.$$

Hence, 2^{2^n+n} must be a perfect square, which implies that $2^n + n$ is even, i.e., n is even. However, as we proved earlier, $\frac{2^n}{n+1}$ is a perfect square only for odd n, which completes the proof. ■

Problem 2. Given an acute triangle ABC. Let D, E, F be the feet of the altitudes from vertices A, B, C respectively. On side BC lies a point K. Let L be the intersection of EF and AK. Prove that the circumcenter of triangle ABC lies on segment AK if and only if $\sphericalangle ELA$ is a right angle.

Author: Tomasz Kossakowski
Solution: Let the intersection of the altitudes be H. Denote $\sphericalangle BAD = \alpha$. The quadrilateral $AFHE$ is cyclic, since

$$\sphericalangle AFH + \sphericalangle AEH = 90^\circ + 90^\circ = 180^\circ.$$

Then

$$\sphericalangle BAD = \sphericalangle FAH = \sphericalangle FEH = \alpha.$$

The points O and H are isogonal conjugates, so O lies on AK if and only if

$$\sphericalangle BAH = \sphericalangle OAC = \alpha.$$

In this case,

$$\sphericalangle LEH = 90^\circ - \alpha,$$

so

$$\sphericalangle ELA = 180^\circ - (90^\circ - \alpha) - \alpha = 90^\circ.$$

■