Problem 1. Prove that if for positive integers a, b, and c the equality

$$\sqrt{a} + \sqrt{b} = c$$

holds, then a and b are perfect squares of integers.

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Solution: Squaring both sides of the given equation we obtain

$$\left(\sqrt{a} + \sqrt{b}\right)^2 = c^2$$

$$a + 2\sqrt{a}\sqrt{b} + b = c^2$$

We know that a, b and c^2 are integers, therefore $2\sqrt{a}\sqrt{b}$ must also be an integer, which means $\sqrt{a}\sqrt{b}$ is rational.
Now let's multiply the equation from the statement by $\sqrt{a}$. Then we obtain $a + \sqrt{b}\sqrt{a} = c\sqrt{a}$. From earlier observations we know that the left-hand side is rational, so the right-hand side must be rational as well. However, if a is not a perfect square of an integer (it cannot be a square of a rational since a is an integer), then $c\sqrt{a}$ is irrational. Therefore, a must be a perfect square of an integer.
An analogous reasoning applies to b, showing that it must also be a perfect square of an integer.
This completes the proof.

Problem 2. Positive numbers x, y, z satisfy the equality $x+y+z=1$. Prove that:

$$\frac{7x^2+4y^2+2z^2}{2(x+y)}+\frac{2x^2+7y^2+4z^2}{2(y+z)}+\frac{4x^2+2y^2+7z^2}{2(z+x)}>3.$$

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Solution: I will show that each of the three fractions on the left-hand side of the inequality is not smaller than 1. I will prove it for the first fraction, the other two are analogous.
So we want to show that

$$\frac{7\cdot x^2+4\cdot y^2+2\cdot z^2}{2\cdot(x+y)}>1$$

Multiplying the inequality on both sides by (positive) $2(x+y)$, we get

$$5\cdot x^2+4\cdot y^2+2\cdot z^2>2\cdot(x+y)$$

Moreover, we have

$$\begin{aligned}2\,(x+y)&=2\,(x+y)\cdot 1=2\,(x+y)\cdot(x+y+z)=\\&=2\left(x^2+y^2+2xy+yz+xz\right)=2x^2+2y^2+4xy+2yz+2xz\end{aligned}$$

Substituting this into the inequality we need to prove, it suffices to show that

$$7\cdot x^2+4\cdot y^2+2\cdot z^2>2x^2+2y^2+4xy+2yz+2xz$$

$$5\cdot x^2+2\cdot y^2+2\cdot z^2-4xy-2yz-2xz>0$$

$$4\cdot x^2-4xy+y^2+y^2-2yz+z^2+x^2-2xz+z^2>0$$

$$(2x-y)^2+(y-z)^2+(x-z)^2>0$$

The weak inequality ($\geqslant$) obviously holds, since the square of a real number is not less than 0. Moreover, for equality to hold, each square, and hence each expression squared, must equal 0. From the second and third square it follows that $y=z=x$, which together with the assumption $x+y+z=1$ gives $x=y=z=\frac{1}{3}$. However, this contradicts the first square, which requires $y=2x$. Therefore, equality cannot hold and we are left with a strict inequality. Hence this part of the inequality has been proven.
Analogously, we prove it for the other two fractions, and summing them up gives the desired inequality.
This completes the proof.