Problem 1. Find the number of ways to distribute n indistinguishable fruits among k children assuming that each child must receive at least two.

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Solution: Let x_i denote the number of fruits received by the i-th child. We have

$$x_1 + \cdots + x_k = n, \qquad x_i \in \mathbb{Z}_{\geqslant 2} \ \ (i = 1, \dots, k).$$

Introduce new variables $y_i = x_i - 2$. Then $y_i \geqslant 0$ and

$$y_1 + \cdots + y_k = n - 2k.$$

The number of nonnegative integer solutions of the above equation (the classical "stars and bars" solution) equals

$$\binom{(n-2k)+k-1}{k-1} = \binom{n-k-1}{k-1},$$

provided that $n - 2k \geqslant 0$ (otherwise there are no solutions).

$$\boxed{\text{For } n \geqslant 2k \ \text{ the number of ways} = \binom{n-k-1}{k-1}. \qquad \text{For } n < 2k \text{ — } 0.}$$

Problem 2. Prove that the equation

$$y^2 + x = \frac{x^2}{2}$$

has infinitely many solutions in integers x, y.

Author: Robert Rośczak
Solution:

Dowód. Consider the equation

$$y^2 + x = \frac{x^2}{2}.$$

Multiplying by 2 we obtain

$$x^2 - 2x - 2y^2 = 0,$$

hence

$$x = 1 \pm \sqrt{1 + 2y^2}.$$

For x to be an integer, the number $1 + 2y^2$ must be a perfect square. Let $t^2 = 1 + 2y^2$. Then

$$t^2 - 2y^2 = 1.$$

This is a Pell equation. It is known to have infinitely many integer solutions (t, y).

These solutions can be generated by the identity

$$t_n + y_n\sqrt{2} = (3 + 2\sqrt{2})^n, \qquad n = 0, 1, 2, \ldots,$$

which yields the recurrence

$$t_{n+1} = 3t_n + 4y_n, \qquad y_{n+1} = 2t_n + 3y_n,$$

with the initial condition $(t_0, y_0) = (1, 0)$.

Since $x = 1 + t$, for every n a solution of the original equation is given by

$$(x_n, y_n) = (1 + t_n,\ y_n).$$

Combining the above formulas we obtain a direct recurrence for the pairs (x_n, y_n):

$$x_{n+1} = 3x_n + 4y_n - 2, \qquad y_{n+1} = 2x_n + 3y_n - 2,$$

with the initial condition $(x_0, y_0) = (2, 0)$.

For example:

$$(x_0, y_0) = (2, 0), \quad (x_1, y_1) = (4, 2), \quad (x_2, y_2) = (14, 10), \quad (x_3, y_3) = (52, 36), \ldots$$

Thus we obtain infinitely many integer solutions of the equation

$$y^2 + x = \frac{x^2}{2}.$$

$\square$