Problem 1. If N is a positive integer, how many integers lie between $\sqrt{N^2+N+1}$ and $\sqrt{9N^2+N+1}$?

Source of the problem: Kangaroo Mathematics 2022, Student level
Selection and editing of solution: Maja Chlewicka
Solution: Since neither $\sqrt{N^2+N+1}$ nor $\sqrt{9N^2+N+1}$ is an integer, we need to find the integers between which these numbers are located.

Starting with $\sqrt{N^2+N+1}$: It is clearly greater than N, and less than $N+1$, which can also be written as $\sqrt{N^2+2N+1}$. Thus the number $\sqrt{N^2+N+1}$ lies between N and $N+1$.

Next, consider $\sqrt{9N^2+N+1}$. It is immediately visible that this number is greater than $2N+1$, since

$$\sqrt{4N^2+4N+1} < \sqrt{9N^2+N+1},$$

which after transformation is equivalent to

$$5N^2 - 3N > 0,$$

which is true for all positive integers N.

Also, $\sqrt{9N^2+N+1}$ is smaller than $3N+1$:

$$\sqrt{9N^2+6N+1} > \sqrt{9N^2+N+1}.$$

Therefore, $\sqrt{9N^2+N+1}$ lies between $2N+1$ and $3N+1$.

Gathering all the information, we see that the integers between the two given numbers are: $N+1$ and $N+2$. So there are 2 integers.

Answer: 2.

Problem 2. Knowing that:

$$\begin{cases} a+b+c=6 \\ a^2+b^2+c^2=14 \end{cases}$$

find

$$(a-b)^2+(b-c)^2+(c-a)^2.$$

Problem author: Antonina Pajek
Solution:

$$(a-b)^2+(b-c)^2+(c-a)^2 = 2(a^2+b^2+c^2-(ab+bc+ca)) \tag{1}$$

Let us find $ab+bc+ca$: from the identity $(a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ca$ we know that:

$$36 = 14 + 2(ab+bc+ca)$$
$$2(ab+bc+ca) = 22$$
$$ab+bc+ca = 11$$

Substituting into (1):

$$(a-b)^2+(b-c)^2+(c-a)^2 = 2\cdot(14-11) = 6.$$