Problem 1. The real numbers x and y satisfy the inequality $x^2 + x \leqslant y$. Prove that $y^2 + y \geqslant x$.

Source: XIV Olimpiada Matematyczna Juniorów Zawody drugiego stopnia
Choice: Maja Chlewicka
Solution: We know that

$$x^2 + y^2 \geqslant 0$$

Transforming the inequality $x^2 + x \leqslant y$:

$$x^2 + x - y \leqslant 0 \quad / \cdot (-1)$$

$$-x^2 - x + y \geqslant 0$$

We combine the two inequalities $x^2 + y^2 \geqslant 0$ i $-x^2 - x + y \geqslant 0$

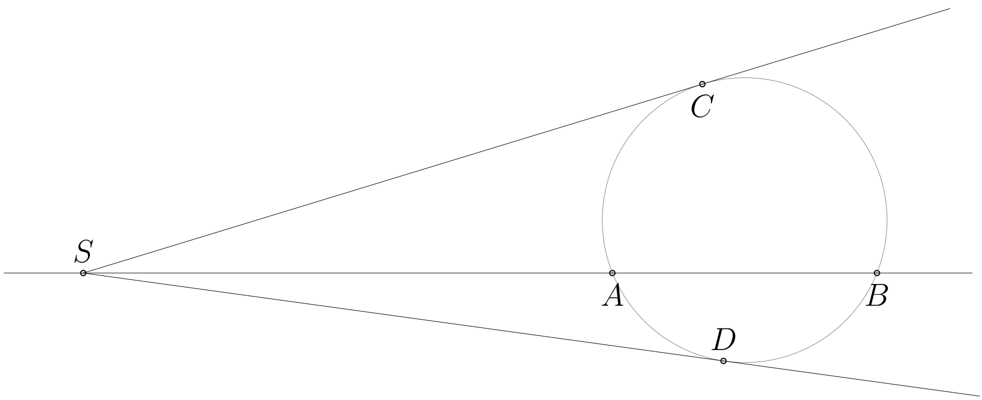
$$x^2 + y^2 - x^2 - x + y \geqslant 0$$

After rearrangement we obtain the result $y^2 + y \geqslant x$ which completes the proof.

Problem 2. Given triangle ABC. The line tangent to the circumcircle of this triangle drawn from point C intersects line AB at point S. Prove that

$$\frac{AS}{SB} = \left(\frac{AC}{CB}\right)^2.$$

Source: Math camp in Poręba Wielka
Choice: Antonina Pajek
Solution:



Let's make a drawing and draw the tangent to the circle at point D. From the theorem on equal tangents we know that $SD = SC$, and because the lines SC and SD are tangent I know that $\sphericalangle SDA = \sphericalangle ABD = \alpha$. From the power of the point S we know that

$$SC^2 = SA \cdot SB = SD^2,$$

i.e.

$$SC^2 = SA \cdot SB \quad / : SB^2$$

we obtain

$$\left(\frac{SC}{SB}\right)^2 = \frac{SA}{SB}.$$

From Ptolemy's theorem we know that

$$AD \cdot CB = DB \cdot AC$$

i.e.

$$\frac{AC}{CB} = \frac{AD}{DB},$$

so the claim can be written as

$$\frac{AS}{SB} = \left(\frac{AC}{CB}\right)^2 = \left(\frac{AD}{DB}\right)^2.$$

Returning to the power of a point, we know that $SB = \frac{SD^2}{SA}$, hence

$$\frac{AS}{\frac{SD^2}{SA}} = \frac{AS^2}{SD^2}.$$

Simplifying the claim we get:

$$\left(\frac{AS}{SD}\right)^2 = \left(\frac{AD}{DB}\right)^2 \Longrightarrow \frac{AS}{SD} = \frac{AD}{DB}.$$

Since $\sphericalangle SDA = \sphericalangle ABD = \alpha$, let $\sphericalangle DAB = \beta$, i.e. $\sphericalangle SAD = 180^\circ - \beta$. Thus we may write the Law of Sines for triangles SAD and DAB. Hence

$$\frac{SA}{\sin\alpha} = \frac{SD}{\sin(180^\circ - \beta)} = \frac{SD}{\sin\beta} \Longrightarrow \frac{SA}{SD} = \frac{\sin\alpha}{\sin\beta},$$

and similarly

$$\frac{AD}{\sin\alpha} = \frac{DB}{\sin\beta} \Longrightarrow \frac{AD}{DB} = \frac{\sin\alpha}{\sin\beta}.$$

Therefore $\frac{SA}{SD} = \frac{AD}{DB}$, which completes the proof.