Problem 1. Prove that

$$\frac{1}{\sqrt[3]{1}}+\frac{1}{\sqrt[3]{2}}+\ldots+\frac{1}{\sqrt[3]{2025}}>200.$$

Problem author: Tomasz Kossakowski
Solution:
Method 1:
Let $x_k=\sqrt[3]{k}$ for $k=1,2,\ldots,2025$. First, we apply the Cauchy inequality in Engel's form:

$$\sum_{k=1}^{2025}\frac{1}{x_k}=\sum_{k=1}^{2025}\frac{1^2}{x_k}\geqslant\frac{\left(\sum_{k=1}^{2025}1\right)^2}{\sum_{k=1}^{2025}x_k}=\frac{2025^2}{\sum_{k=1}^{2025}x_k}.$$

Next, we use the power mean inequality for powers $p=1$ and $q=3$ (for positive numbers and $p<q$ we have $M_p\leqslant M_q$), in the form

$$\left(\frac{1}{n}\sum_{k=1}^{n}x_k\right)^3\leqslant\frac{1}{n}\sum_{k=1}^{n}x_k^3.$$

For $x_k=\sqrt[3]{k}$ and $n=2025$, this gives

$$\left(\frac{1}{2025}\sum_{k=1}^{2025}\sqrt[3]{k}\right)^3\leqslant\frac{1}{2025}\sum_{k=1}^{2025}(\sqrt[3]{k})^3=\frac{1}{2025}\sum_{k=1}^{2025}k.$$

Hence

$$\sum_{k=1}^{2025}\sqrt[3]{k}\leqslant 2025\left(\frac{1}{2025}\sum_{k=1}^{2025}k\right)^{1/3}.$$

Compute the arithmetic sum:

$$\sum_{k=1}^{2025}k=\frac{2025\cdot(2025+1)}{2}=\frac{2025\cdot 2026}{2}=2025\cdot 1013.$$

Thus,

$$\sum_{k=1}^{2025}\sqrt[3]{k}\leqslant 2025\cdot 1013^{1/3}.$$

Returning to Engel's inequality:

$$\sum_{k=1}^{2025}\frac{1}{\sqrt[3]{k}}\geqslant\frac{2025^2}{\sum_{k=1}^{2025}\sqrt[3]{k}}\geqslant\frac{2025^2}{2025\cdot 1013^{1/3}}=\frac{2025}{1013^{1/3}}>200.$$

Therefore,

$$\sum_{k=1}^{2025} \frac{1}{\sqrt[3]{k}} > 200.$$

Method 2:
The function $f(x) = x^{-1/3}$ is decreasing, so

$$\sum_{n=1}^{2025} \frac{1}{\sqrt[3]{n}} > \int_1^{2026} \frac{1}{\sqrt[3]{x}}\,dx = \frac{3}{2}x^{2/3}\Big|_1^{2026} = \frac{3}{2} \cdot 2026^{2/3} - \frac{3}{2} > 238.$$

Problem 2. Let σ be a composite number. Suppose the equation

$$\frac{1}{a}+\frac{1}{b}=\frac{1}{\sigma}$$

is satisfied by some positive integers a, b. Show that there exist three pairs $(a_1, b_1), (a_2, b_2), (a$
of distinct positive integers, different from (a, b), which also satisfy this equation.

Problem author: Bartosz Trojanowski
Solution: We prove that all triples (a, b, σ) satisfying the equation are of the form $a = kx(x + y), b = ky(x + y), \sigma = kxy$.
Multiplying both sides by $ab\sigma$ gives

$$\sigma(a+b) = ab \iff (a-\sigma)(b-\sigma) = \sigma^2.$$

If $\gcd(a, b, \sigma) > 1$, we can divide the equation by it since both sides are of the same degree. Therefore, without loss of generality, assume $\gcd(a, b, \sigma) = 1$. Clearly, $\sigma > 1$, so it has a prime divisor p. Then

$$2v_p(\sigma) = v_p((a-\sigma)(b-\sigma)) = v_p(a-\sigma) + v_p(b-\sigma).$$

If both numbers are divisible by p, then $p \mid a, b, \sigma$, which contradicts the coprimality assumption. Hence the p-adic exponent of one of $a-\sigma$ or $b-\sigma$ equals $2v_p(\sigma)$. Repeating for each prime divisor, we obtain that $a - \sigma$ and $b - \sigma$ are perfect squares. So $a - \sigma = x^2, b - \sigma = y^2$ for some x, y. Then

$$\sigma^2 = x^2y^2 \implies \sigma = xy.$$

Hence

$$a = x(x+y), \quad b = y(x+y).$$

We can also multiply a, b, σ by some k, which was previously excluded by the gcd.

It remains to show that any composite σ can be represented as kxy in at least 4 ways. Since σ is composite, it has two distinct divisors $p, q > 1$. Let $\sigma = pqA$ (where A may be 1). Then we can take the triples:

$$(k, x, y) = (pq, A, 1), (p, A, q), (q, A, p), (1, Ap, q).$$

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