Problem 1. A swimmer was moving *upstream* with a speed of $v = 10\frac{\text{km}}{\text{h}}$. Then he turned back and covered the same distance swimming *downstream*. Show that on the whole trip (up and down the river) he will not achieve an average speed of $20\frac{\text{km}}{\text{h}}$.

Author: Michał Fronczek
Solution: Let t_1 denote the time it took the swimmer to cover the distance upstream, and let s be the length of that distance. Let t_2 denote the time of the return journey, with speed v_2. Since the distance in both directions is the same, we have $t_1 \cdot v = s = t_2 \cdot v_2$. From the definition of average speed we get:

$$v_{\text{avg}} = \frac{s_c}{t_c} = \frac{s+s}{t_1+t_2} = \frac{2 \cdot t_1 \cdot v}{t_1+t_2}$$

Of course, teleportation is not allowed, since it was stated that the swimmer swam back, so $t_2 > 0$ and $t_1 + t_2 > t_1$. Hence, returning to our expression for the average speed, we have:

$$v_{\text{avg}} = \frac{2 \cdot t_1 \cdot v}{t_1+t_2} < \frac{2 \cdot t_1 \cdot v}{t_1} = 2 \cdot v = 20\frac{\text{km}}{\text{h}}$$

Therefore, it is impossible to achieve this speed, since any actual average speed will always be smaller.

Problem 2. Let there be a sequence of natural numbers and a prime number p such that for all $i \in \mathbb{Z}_+$ the recurrence relation

$$a_{i+1} = a_i^p + 1$$

holds. Prove that regardless of the choice of the initial term a_1 and the prime p, the sequence will always contain a composite number.

Author: Michał Fronczek
Solution: Let us recall one important theorem first:

Fermat's Little Theorem: For any positive integer a and any prime number p, the following holds:

$$a^p \equiv a \pmod{p}$$

Now, let us return to the problem. Consider the sequence modulo p. By the above theorem, we have $a_{i+1} = a_i^p + 1 \equiv a_i + 1 \pmod{p}$. Considering k successive recurrences, we get:

$$a_{i+k} \equiv a_{i+k-1} + 1 \equiv a_{i+k-2} + 1 + 1 \equiv \ldots \equiv a_i + k \pmod{p}$$

Let $a_1 \equiv n \pmod{p}$. Then, for $k = l \cdot p - n$, we obtain:

$$a_{1+k} \equiv a_1 + k \equiv n + l \cdot p - n \equiv 0 \pmod{p}$$

Thus, p divides a_{1+k}, and moreover $k = l \cdot p - n$ can be arbitrarily large, while the sequence—by its recursive definition—is strictly increasing. This means that its terms can become arbitrarily large, and every p-th term is divisible by p. For such a term to be prime, it would have to be exactly equal to p, which contradicts the monotonicity of the sequence. Hence, not all such terms are equal to p, but all are divisible by p, and therefore composite.

This reasoning holds for any choice of a_1 and p. This completes the proof.