Problem 1. At a meeting, there are 6 people. Some pairs of them are friends. Prove that there exists a group of three people such that either all of them know each other, or none of them do.

Source: Well-known problem
Selection: Robert Rościzak
Solution: Consider the people as vertices of a complete graph. Color an edge blue if the two people connected by it are friends, and red otherwise. For a certain vertex A, by the Pigeonhole Principle, there are at least 3 edges of the same color emanating from it. Denote the vertices at the ends of these edges by X, Y, Z. If none of these three people know each other, then (X, Y, Z) satisfies the required condition. Otherwise, some pair among them are friends, and that pair together with A satisfies the condition.

Questions of this type belong to a branch of mathematics known as *Ramsey Theory*. In this case, we can use the fact that follows from Ramsey's Theorem, namely that $R(3, 3) = 6$, meaning that a two-colored complete graph must have at least 6 vertices to ensure the existence of a red triangle or a blue triangle.

Problem 2. Let $ABCD$ be a parallelogram and M the intersection point of its diagonals. Prove that the perpendicular projections of point B onto the lines AC, CD, and AD, together with point M, are concyclic.

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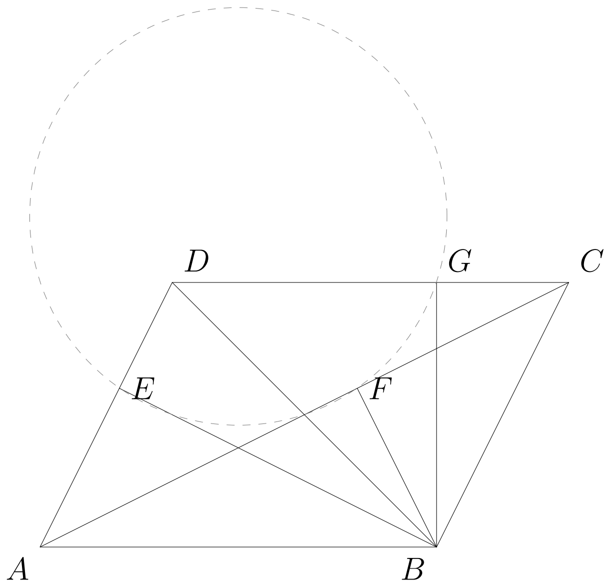
Solution:



Solution 1. Let the perpendicular projections of point B onto the lines AD, AC, and CD be denoted respectively by E, F, G.

Notice that triangle BMG is isosceles, since M lies on the perpendicular bisector of any altitude of the parallelogram, in particular also on BG. Similarly, triangle BME is isosceles for the same reason — M lies on the perpendicular bisector of BE. Therefore, $MB = MG = ME$, so point M is equidistant from the vertices of triangle EBG, meaning it is the circumcenter of that triangle.

Since $\sphericalangle AEB = \sphericalangle AFB = 90°$, quadrilateral $ABFE$ is cyclic.

Let $\sphericalangle EFA = \alpha$. Then $\sphericalangle EFM = 180° - \alpha$. Moving along the circle, $\sphericalangle ABE = \alpha$. Since $\sphericalangle AEB = 90°$, we have $\sphericalangle EAB = 90° - \alpha$. From the properties of the parallelogram and right angles, we obtain successively: $\sphericalangle BCG = 90° - \alpha$, $\sphericalangle CBG = \alpha$, and $\sphericalangle EBG = 90° - \alpha$. The central angle subtending arc EG is twice the inscribed one, hence $\sphericalangle EMG = 2\sphericalangle EBG = 180° - 2\alpha$. Therefore, in isosceles triangle EMG we have $\sphericalangle MEG = \sphericalangle MGE = \alpha$.

Since $\sphericalangle EFM = 180^\circ - \alpha$ and $\sphericalangle MGE = \alpha$, their sum equals 180°, which proves that quadrilateral $GEFM$ is cyclic.

Solution 2. Let the notation of points remain the same as in the first solution.

Construct a circle circumscribed about triangle ACD.

Note that if line AC is the diameter of the circle, then $\sphericalangle ADC = 90°$, and thus $ABCD$ is a square. In this case, the projection of B onto AC coincides with the intersection point of the diagonals, so we need to prove the concyclicity of three points, which trivially holds.

Assume therefore that AC is not the diameter of the circle. Let the tangents to the circle at points A and C meet at point R (they certainly intersect since we excluded the case where they would be parallel).

Let $\sphericalangle RAD = \alpha$. By the theorem on the angle between a tangent and a chord, we have $\sphericalangle ACD = \sphericalangle BAC = \alpha$. Hence, points R and B are *isogonally conjugate* in angle $\sphericalangle CAD$. Similarly, let $\sphericalangle RCD = \beta$, which gives $\sphericalangle CAD = \sphericalangle BAC = \beta$. Thus, points R and B are also isogonally conjugate in angle $\sphericalangle ACD$.

By the definition of isogonal conjugation, if a pair of points is isogonally conjugate in two angles of triangle ACD, then it is isogonally conjugate in the entire triangle.

A known fact states that the perpendicular projections of isogonally conjugate points in a triangle onto its sides lie on a single circle. We know that points E, F, G are the perpendicular projections of B onto the sides of triangle ACD; it remains to show that M is the projection of R onto side AC.

However, since the tangent segments from a point to a circle are equal, we have $RA = RC$, meaning that triangle ARC is isosceles. From the definition of a parallelogram, M is the midpoint of segment AC. In an isosceles triangle, the altitude from the vertex between the equal sides passes through the midpoint of the base — in our case, point M. This altitude coincides with the perpendicular projection, hence point M is indeed the projection of R, completing the proof.

Solution 3. Let the perpendicular projections onto lines AD, AC, and CD be denoted respectively by E, F, G. We apply an inversion in the circle with diameter BD. The claim is equivalent to stating that the images of E, F, G under this inversion lie on a straight line.

Note that $\sphericalangle BED = \sphericalangle BGD = 90°$, hence points E and G lie on the circle of inversion. It remains to prove that the image of F lies on the line through them. We use the *Simson Line Theorem* for this.

Define point K on line AD such that $F'K \perp BF'$. Let L and N be the intersections of line $F'K$ with CD and AB, respectively. To prove the claim, we must show that points K, L, D, A are concyclic. By Thales' theorem,

$$1 = \frac{|BM|}{|MD|} = \frac{|NF'|}{|KF'|},$$

which implies $\sphericalangle BKF' = \sphericalangle BNF' = 180° - \sphericalangle DBN = \sphericalangle BDG$, establishing the desired concyclicity.