Problem 1. Prove that for any prime number $p > 3$ the following holds

$$p^2 \;\Big|\; \binom{p^2}{p} - \binom{p}{1}.$$

Bonus: Prove that the above expression is divisible by a higher power of p. By which highest one?

Author: Robert Rościczak **Solution:** Consider the polynomial

$$F(x) = x(x-1)\dots(x-p+2).$$

Note that:

$$\binom{p^2}{p} - \binom{p}{1} = \frac{p^2(p^2-1)\dots(p^2-p+1)}{p(p-1)\dots\cdot 1} - p = p \cdot \left(\frac{F(p^2-1)-F(p-1)}{F(p-1)}\right)$$

Thus it remains to prove that the expression in the parentheses is divisible by p. Note that. Recall the well-known lemma that for any integer polynomial P and integers a, b we have $a - b \mid P(a) - P(b)$. In our case $p \mid p^2 - p \mid F(p^2-1) - F(p-1)$. Moreover $p \nmid F(p-1)$, so p divides the expression in the parentheses, which was to be shown.

Problem 2. Prove, that for $a, b, c \in [0, 1]$ the following inequality is true

$$\sqrt{abc} + \sqrt{(1-a)(1-b)(1-c)} \leqslant 1.$$

Author: Robert Rośczak
Solution:

Solution 1. Substitute $a = \sin^2 \alpha, b = \sin^2 \beta, c = \sin^2 \gamma$ for angles $\alpha, \beta, \gamma \in [0°, 90°]$. We can transform

$$\sqrt{\sin^2 \alpha \sin^2 \beta \sin^2 \gamma} + \sqrt{(1-\sin^2 \alpha)(1-\sin^2 \beta)(1-\sin^2 \gamma)} \leqslant 1.$$

From the trigonometric identity we substitute

$$\sqrt{\sin^2 \alpha \sin^2 \beta \sin^2 \gamma} + \sqrt{\cos^2 \alpha \cos^2 \beta \cos^2 \gamma} \leqslant 1,$$

$$|\sin \alpha \sin \beta \sin \gamma| + |\cos \alpha \cos \beta \cos \gamma| \leqslant 1,$$

$$\sin \alpha \sin \beta \sin \gamma + \cos \alpha \cos \beta \cos \gamma \leqslant 1.$$

We know that $\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$. Consider two cases:

a) when $\sin \gamma \leqslant \cos \gamma$.

$$\sin \alpha \sin \beta \sin \gamma + \cos \alpha \cos \beta \cos \gamma \leqslant \sin \alpha \sin \beta \cos \gamma + \cos \alpha \cos \beta \cos \gamma =$$

$$= \cos \gamma(\sin \alpha \sin \beta + \cos \alpha \cos \beta) = \cos \gamma \cos(\alpha - \beta) \leqslant 1 \cdot 1 = 1.$$

b) when $\cos \gamma < \sin \gamma$.

$$\sin \alpha \sin \beta \sin \gamma + \cos \alpha \cos \beta \cos \gamma < \sin \alpha \sin \beta \sin \gamma + \cos \alpha \cos \beta \sin \gamma =$$

$$= \sin \gamma(\sin \alpha \sin \beta + \cos \alpha \cos \beta) = \sin \gamma \cos(\alpha - \beta) \leqslant 1 \cdot 1 = 1.$$

Thus the inequality holds for any $a, b, c \in [0, 1]$.

Solution 2. Since $abc \in [0, 1]$, the inequality $\sqrt{abc} \leqslant \sqrt[3]{abc}$ holds. We can therefore estimate (similarly for the second product):

$$\sqrt{abc} + \sqrt{(1-a)(1-b)(1-c)} \leqslant \sqrt[3]{abc} + \sqrt[3]{(1-a)(1-b)(1-c)}.$$

Now we can invoke twice the inequality between the geometric and arithmetic means:

$$\sqrt[3]{abc} + \sqrt[3]{(1-a)(1-b)(1-c)} \leqslant \frac{a+b+c}{3} + \frac{1-a+1-b+1-c}{3} = 1,$$

which was to be proved.