Problem 1. The functions f and g are defined on the set of real numbers and satisfy the system of equations

$$\begin{cases} f(x) + 2g(1-x) = x^2 \\ f(1-x) - g(x) = x^2 \end{cases} \tag{1}$$

By what formula is $f(x)$ expressed?

Source: Międzynarodowy Konkurs Matematyczny KANGUR 2023 Student
Choice: Maja Chlewicka
Solution: We substitute $1-x$ for x in the first equation

$$f(1-x) + 2g(1-(1-x)) = (1-x)^2$$
$$f(1-x) + 2g(x) = (1-x)^2$$

Thus our system of equations becomes

$$f(1-x) + 2g(x) = (1-x)^2$$
$$f(1-x) - g(x) = x^2$$

Next we subtract one equation from the other

$$f(1-x) + 2g(x) - f(1-x) + g(x) = (1-x)^2 - x^2$$
$$3g(x) = 1 - 2x + x^2 - x^2$$
$$g(x) = \frac{1-2x}{3}$$

Therefore substituting into the second equation we have:

$$f(1-x) - \frac{1-2x}{3} = x^2$$
$$f(1-x) = x^2 + \frac{1-2x}{3}$$
$$f(1-x) = \frac{3x^2 + 1 - 2x}{3}$$

To find $f(x)$ we substitute x by $1-x$ in the expression above, obtaining:

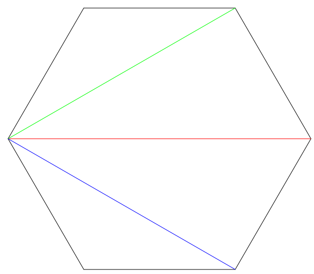
$$f(1-(1-x)) = \frac{3(1-x)^2 + 1 - 2(1-x)}{3}$$
$$f(x) = \frac{3 - 6x + 3x^2 - 1 - 2 + 2x}{3}$$
$$f(x) = \frac{3x^2 - 4x + 2}{3}$$

Answer:

$$f(x) = \frac{3x^2 - 4x + 2}{3}$$

Problem 2. A regular hexagon is given. Tosia and Mateusz play a game consisting of drawing diagonals – alternately they draw diagonals joining two vertices, however the diagonals they draw are not allowed to intersect. The player who cannot make a move loses. Tosia starts the game; determine who will win the game and what the winning strategy is.

Author of the problem: Antonina Pajek
Solution: We will show that Tosia wins the game and determine her winning strategy. Let us draw a regular hexagon. Since Tosia starts the game, she can draw any diagonal, so to win she should draw the diagonal marked in red in the diagram below:



Next, Mateusz draws any diagonal (of course, one that does not intersect the red diagonal drawn earlier by Tosia). Let us assume that the diagonal drawn by Mateusz is the blue one.

Notice that if Mateusz can draw a diagonal (which does not give Tosia an immediate win), then Tosia is guaranteed to have a move as well — it is enough for her to mirror Mateusz's move with respect to the red diagonal (for example, after Mateusz draws the blue diagonal, Tosia draws the green diagonal).

Therefore, we see that whenever Mateusz has a possible move, Tosia also has one. However, since the players can only make a finite number of moves (because a hexagon has a finite number of diagonals), eventually Mateusz will be the first who cannot move — and thus he loses. Hence, Tosia always wins.