Problem 1. Find the number of pairs (a, b) smaller than their respective moduli for which the system of congruences

$$\begin{cases} x \equiv a \pmod{15}, \\ x \equiv b \pmod{12} \end{cases}$$

has a solution.

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Solution: Let us rewrite the given system of congruences in an equivalent form:

$$\begin{cases} x \equiv a \pmod{5}, \\ x \equiv a \pmod{3}, \end{cases} \qquad \begin{cases} x \equiv b \pmod{4}, \\ x \equiv b \pmod{3}. \end{cases}$$

We see that for the system to have a solution, it is necessary and sufficient that

$$a \equiv b \pmod{3}.$$

Let us now count how many numbers smaller than 12 and 15 give each remainder modulo 3.

For numbers modulo 12:

$$\begin{aligned} &\text{remainder } 0 : \{0, 3, 6, 9\} \quad (4 \text{ numbers}), \\ &\text{remainder } 1 : \{1, 4, 7, 10\} \quad (4 \text{ numbers}), \\ &\text{remainder } 2 : \{2, 5, 8, 11\} \quad (4 \text{ numbers}). \end{aligned}$$

For numbers modulo 15:

$$\begin{aligned} &\text{remainder } 0 : \{0, 3, 6, 9, 12\} \quad (5 \text{ numbers}), \\ &\text{remainder } 1 : \{1, 4, 7, 10, 13\} \quad (5 \text{ numbers}), \\ &\text{remainder } 2 : \{2, 5, 8, 11, 14\} \quad (5 \text{ numbers}). \end{aligned}$$

For a fixed remainder modulo 3, we can therefore choose a pair (a, b) in $4 \cdot 5 = 20$ ways. Since there are three possible remainders modulo 3, the total number of pairs (a, b) equals

$$3 \cdot 20 = 60.$$