Problem 1. Let $a, b, c > 0$ and $a + b + c = 67$. Prove that

$$4\left(a\sqrt[3]{a} + b\sqrt[3]{b} + c\sqrt[3]{c}\right) + a + b + c \geqslant 1000.$$

Problem Author: Tomasz Kossakowski
Solution: Consider the function

$$f(x) = x^{4/3}, \quad x > 0.$$

We have

$$f''(x) = \frac{4}{9}x^{-2/3} > 0,$$

so the function f is convex on $(0, \infty)$.

By Jensen's inequality for three variables, we get

$$\frac{f(a) + f(b) + f(c)}{3} \geqslant f\left(\frac{a + b + c}{3}\right).$$

Hence,

$$a^{4/3} + b^{4/3} + c^{4/3} \geqslant 3\left(\frac{67}{3}\right)^{4/3}.$$

Substituting into the left-hand side of the inequality from the problem statement, we obtain

$$4\left(a^{4/3} + b^{4/3} + c^{4/3}\right) + 67 \geqslant 4 \cdot 3\left(\frac{67}{3}\right)^{4/3} + 67.$$

Calculating the right-hand side gives

$$4 \cdot 3\left(\frac{67}{3}\right)^{4/3} + 67 \approx 1147 > 1000.$$

Thus,

$$4\left(a\sqrt[3]{a} + b\sqrt[3]{b} + c\sqrt[3]{c}\right) + a + b + c \geqslant 1000,$$

which concludes the proof. □

Problem 2. Define the sequence a_n by: $a_1 = 1, a_{n+1} = a_n + n + 1$ for every positive integer n. A positive integer k is called *Chinese-friendly* if the number $8k + 1$ is a perfect square and

$$\prod_{i=1,\ a_i \leqslant k} a_i$$

is a perfect square. Prove that there are infinitely many Chinese-friendly numbers.

Problem Author: Bartosz Trojanowski
Solution: Let us choose k such that $\sqrt{8k+1} \in \mathbb{Z}$. First, we show that there exists a number s such that $a_s = k$. Notice that $a_n = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$. The number $8k + 1$ is odd, so let us write $8k + 1 = (2t + 1)^2$. Simple calculations show that $k = \frac{t(t+1)}{2}$. We can take $s = t$. Then

$$\prod_{i=1,\ a_i \leqslant k} a_i = \prod_{i=1}^{t} a_i = \frac{1 \cdot 2}{2} \cdot \frac{2 \cdot 3}{2} \cdot \cdots \cdot \frac{t(t+1)}{2} = \frac{(t!)^2(t+1)}{2^t}.$$

It remains to show that there are infinitely many t such that $\frac{t+1}{2^t}$ is a perfect square of a rational number. Let $t + 1 = 2x^2$. Then $\frac{t+1}{2^t} = \left(\frac{x}{2^{x^2-1}}\right)^2$, so this number is indeed a perfect square. Since $8\frac{2x^2(2x^2-1)}{2} + 1 = (4x^2 - 1)^2$, the conditions of the problem are satisfied. ■

With this set of problems, we close not only the year 2025 but also the series of problems regularly published by our project.

Thank you all for your participation, time, and mathematical engagement.

MathLovers Team

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